Lecture 05

Differential Kinematics

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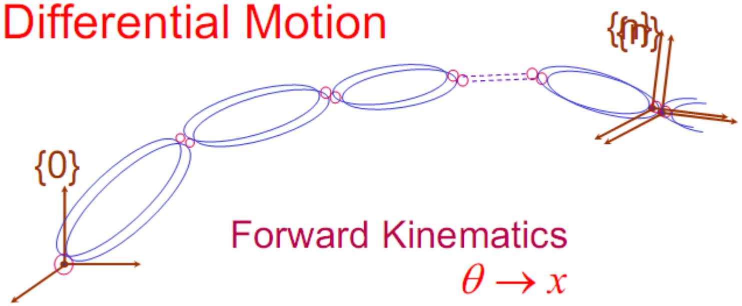
Guiding Question

- In robotic applications, not only the position and orientation, but the velocity of the end-effecter is also to be monitored and controlled.
- How can the velocity of the end-effector be calculated?
- In order to move the end-effecter in a specified direction with a specified speed, it is necessary to coordinate the speeds of the individual joints
- Fundamental methods are to be developed for achieving such coordinated joint motion in multiple-joint robotic systems.
- We derive the differential relationship between the joint displacements and the end-effecter location, and then solve for the individual joint motions.

Differential Relationship

Differential Motion



Forward Kinematics

$$\theta \rightarrow x$$

Instantaneous Kinematics

$$\theta + \delta\theta \rightarrow x + \delta x$$

Relationship: $\boxed{\delta\theta \leftrightarrow \delta x}$

$\boxed{\dot{\theta} \leftrightarrow \dot{x}}$ → Linear Velocity, Angular Velocity

Generalized Co-ordinate, q

Joint Coordinates

$$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \ldots . q_n)^T}$

Jacobian: Direct Differentiation

- End-effector position **x**
- End-effector diflection δ**x**

$$x = f(q); \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix}$$

$$\delta x_1 = \frac{\partial f_1}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_1}{\partial q_n}\delta q_n$$

$$\vdots$$

$$\delta x_m = \frac{\partial f_m}{\partial q_1}\delta q_1 + \cdots + \frac{\partial f_m}{\partial q_n}\delta q_n$$

$$\delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} . \delta q$$

$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\, \delta q_{(n\times 1)}$$

$$J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)$$

Jacobian: Direct Differentiation

- Due to a “small movements” of individual joints at the current position δ***q***, the resultant motion of the end-effecter is δ**x**. Jacobian matrix (matrix of partial derivatives) relate δ***q*** to δ**x**
- Those small movements are divided by δt to derive the relationship between joint and Cartesian velocities

$$\delta \mathbf{x} = \mathbf{J}\,\delta \mathbf{q}$$

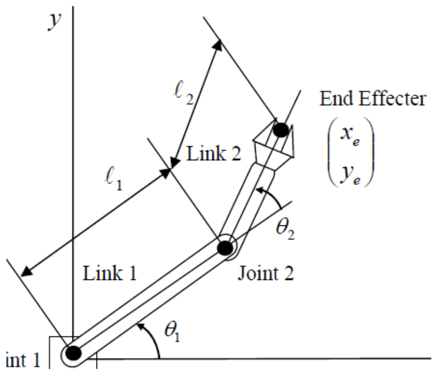
$$\frac{\delta \mathbf{x}}{\delta t} = \mathbf{J}\frac{\delta \mathbf{q}}{\delta t}$$

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$$

$$\dot{\mathbf{x}} = (\dot{x}_e, \dot{y}_e), \dot{\mathbf{q}} = (\dot{\theta}_1, \dot{\theta}_2)$$

Example

- Forward kinematics of the planner manipulator.



$$x_e = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$y_e = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\psi = \theta_1 + \theta_2$$

- The differential relationship is

$$\delta x_e = \frac{\partial x_e}{\partial \theta_1}\delta\theta_1 + \frac{\partial x_e}{\partial \theta_2}\delta\theta_2$$

$$\delta y_e = \frac{\partial y_e}{\partial \theta_1}\delta\theta_1 + \frac{\partial y_e}{\partial \theta_2}\delta\theta_2$$

$$\delta\psi = \delta\theta_1 + \delta\theta_2$$

$$\begin{pmatrix} \delta x_e \\ \delta y_e \\ \delta\psi \end{pmatrix} = \begin{pmatrix} \frac{\partial x_e}{\partial \theta_1} & \frac{\partial x_e}{\partial \theta_2} \\ \frac{\partial y_e}{\partial \theta_1} & \frac{\partial y_e}{\partial \theta_2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \delta\theta_1 \\ \delta\theta_2 \end{pmatrix}$$

$$\delta \mathbf{x} = \quad \mathbf{J} \quad \delta \mathbf{q}$$

Example

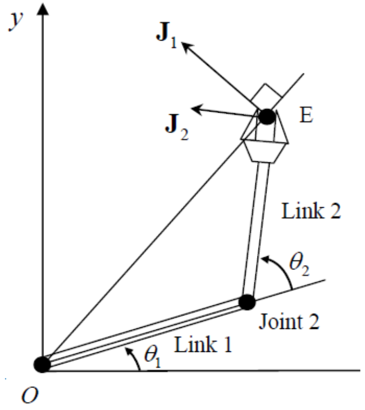
$$\mathbf{J}(\theta_1, \theta_2) = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{J}(\theta_1, \theta_2) = \begin{pmatrix} -y_e & -l_2 \sin(\theta_1 + \theta_2) \\ x_e & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix}$$

Jacobian

- Jacobian provides the relationship between the joint velocities and the resultant end-effecter velocity
- Jacobian can be resolved as follows

$$\mathbf{J} = [[\mathbf{J}_1][\mathbf{J}_2]]$$
$$\dot{\mathbf{x}} = \mathbf{J}_1\dot{\theta}_1 + \mathbf{J}_2\dot{\theta}_2$$



- In general, each column vector of the Jacobian represents the end-effecter velocity and angular velocity generated by the individual joint velocity while all other joints are immobilized

$$\dot{\mathbf{p}} = \mathbf{J}_1\dot{\mathbf{q}}_1 + \cdots + \mathbf{J}_n\dot{\mathbf{q}}_n$$

$$(\dot{x}, \dot{y}, \dot{z}, \dot{\phi}_x, \dot{\phi}_y, \dot{\phi}_z)^T$$

Homework: Joint Velocity Profile

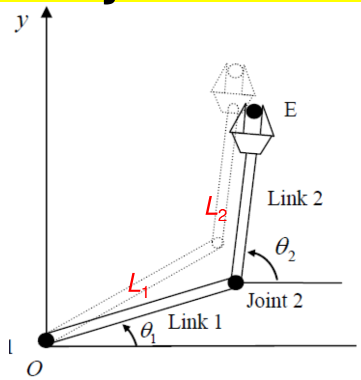
- For the planner two-link manipulator shown, determine the end-effector velocity profile for the following 10s motion

$\boldsymbol{\theta}(0\text{s}) = (0°,0°)^T$

$\boldsymbol{\theta}(5\text{s}) = (45°,90°)^T$

$\boldsymbol{\theta}(10\text{s}) = (90°,0°)^T$

Assume $L_1 = 10$cm, $L_2 = 8$cm

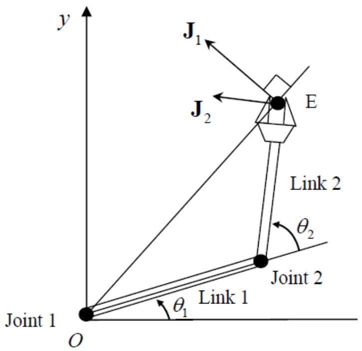


- Write Malab m-file, to draw end-effector speed in x and y directions

Singular Arm Configurations

$$\dot{\mathbf{x}} = \mathbf{J}_1\dot{\theta}_1 + \mathbf{J}_2\dot{\theta}_2$$

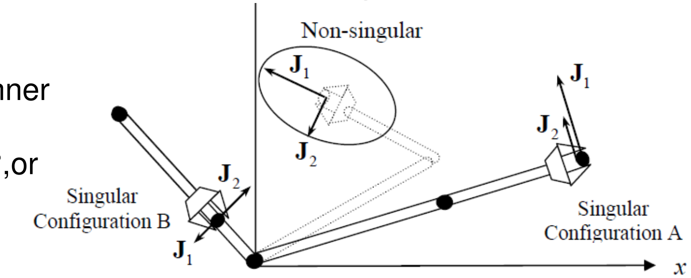
- As long as $\mathbf{J}_1$ and $\mathbf{J}_2$ are not aligned, velocities of the two joints can be set accordingly to make the end-effector move in any direction.



- Directions of $\mathbf{J}_1$ and $\mathbf{J}_2$ are configuration-dependant, and when they are aligned, end-effector is only movable along that direction.
- Such arm configurations are known as singular arm configurations

Singular Arm Configurations

- Singularities occur in planner two-link arm when $\theta_2 = 0°$,or $180°$



$$\mathbf{J}(\theta_1,\theta_2) = \begin{pmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{pmatrix}$$

At singularity configurations

$$\mathbf{J}_{\theta_2=0^o} = \begin{pmatrix} -(l_1+l_2)\sin\theta_1 & -l_2\sin\theta_1 \\ (l_1+l_2)\cos\theta_1 & l_2\cos\theta_1 \end{pmatrix}$$

$$\mathbf{J}_{\theta_2=180^o} = \begin{pmatrix} (-l_1+l_2)\sin\theta_1 & l_2\sin\theta_1 \\ (l_1-l_2)\cos\theta_1 & -l_2\cos\theta_1 \end{pmatrix}$$

Column vectors line up each other

Determinant of J and Singularity

$$\mathbf{J}(\theta_1,\theta_2)=\begin{pmatrix}-l_1\sin\theta_1-l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2)\\ l_1\cos\theta_1+l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{pmatrix}$$

$$\begin{aligned}|\mathbf{J}|&=-l_2\cos(\theta_1+\theta_2)[l_1\sin\theta_1+\cancel{l_2\sin(\theta_1+\theta_2)}]\\&\quad+l_2\sin(\theta_1+\theta_2)[l_1\cos\theta_1+\cancel{l_2\cos(\theta_1+\theta_2)}]\\&=l_1l_2[\sin(\theta_1+\theta_2)\cos\theta_1-\sin(\theta_1+\theta_2)\cos\theta_1]\\&=l_1l_2\sin\theta_2\end{aligned}$$

- At singular arm configurations $\theta_2=0°$. or $\theta_2=180°$

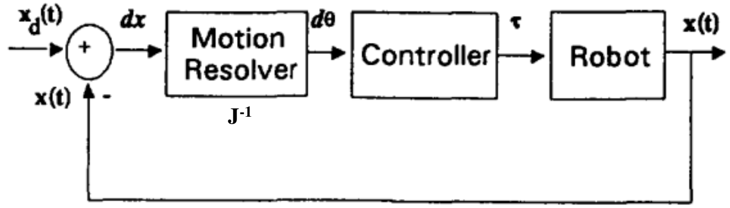
$$|\mathbf{J}|=l_1l_2\sin\theta_2=0$$

Inverse Kinematics of Differential Motion

- Resolve end-effector velocity into velocities of individual joints. Whenever Jacobian is not singular, inverse kinematics can be solved as follows

$$\dot{\mathbf{q}}=\mathbf{J}^{-1}\dot{\mathbf{x}}\quad\text{or}\quad\delta\mathbf{q}=\mathbf{J}^{-1}\delta\mathbf{x}$$

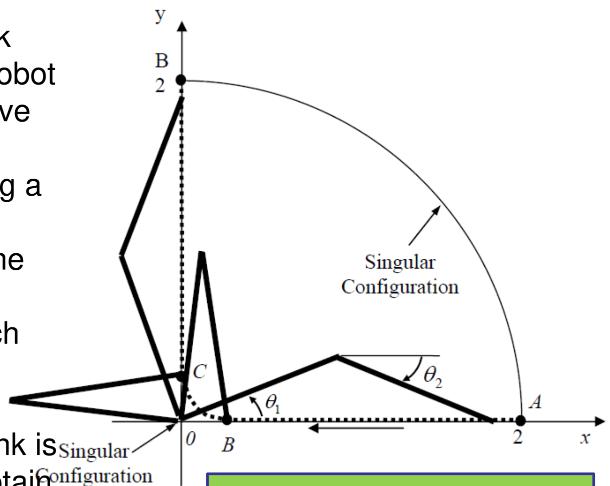
- The solution is unique, unlike the inverse kinematics of end-effector position, where multiple solutions exist.
- This mapping can be used for robot manipulator control as proposed (Resolved Motion Rate Control. Daniel Whitney 1969]



Motion Near Singularities

Consider the two-link planner articulated robot arm. We want to move the endpoint at a constant speed along a path staring at point A(2,0), go close to the origin through B(ε,0) and C(0,ε), and reach the final point D(0,2)

Assume each arm link is of unit length and obtain the profiles of the individual joint velocities.



Comment on joint velocities in B – C segment of the motion

Motion Near Singularities cntd.

- By inverting velocity kinematics

$$\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix}=\mathbf{J}^{-1}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$$

$$\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix}=\begin{pmatrix}-l_1\sin\theta_1-l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2)\\ l_1\cos\theta_1+l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{pmatrix}^{-1}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$$

$$\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix}=\frac{1}{l_1l_2\sin\theta_2}\begin{pmatrix}l_2\cos(\theta_1+\theta_2) & l_2\sin(\theta_1+\theta_2)\\ -l_1\cos\theta_1-l_2\cos(\theta_1+\theta_2) & -l_1\sin\theta_1-l_2\sin(\theta_1+\theta_2)\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$$

$$\dot\theta_1=\frac{\cos(\theta_1+\theta_2)\dot x+\sin(\theta_1+\theta_2)\dot y}{l_1\sin\theta_2}$$

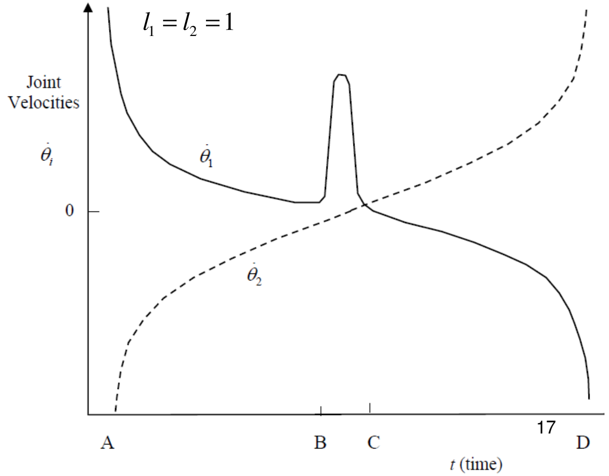
$$\dot\theta_2=\frac{-[l_1\cos\theta_1+l_2\cos(\theta_1+\theta_2)]\dot x-[l_1\sin\theta_1+l_2\sin(\theta_1+\theta_2)]\dot y}{l_1l_2\sin\theta_2}$$

Motion Near Singularities cntd.

- Very high joint velocities are resulted at points A and D, which are the arm's singular configurations (θ_2=0°)
- Close to the origin (θ_2≈–180°), the velocity of the first joint becomes very large in order to quickly turn the arm from B to C,

$$\dot{\theta}_1 = \frac{v_x \cos(\theta_1+\theta_2) + v_y \sin(\theta_1+\theta_2)}{\sin\theta_2}$$

$$\dot{\theta}_1 = \frac{v_x[\cos(\theta_1)+\cos(\theta_1+\theta_2)] + v_y \sin[\theta_1 + \sin(\theta_1+\theta_2)]}{\sin\theta_2}$$



Singularity Analysis

- When the arm is fully extended (θ_2=0°). For position A (θ_1=0°).

$$\mathbf{J}_1 = \begin{pmatrix} -2\sin(\theta_1=0) \\ 2\cos(\theta_1=0) \end{pmatrix} = \begin{pmatrix} 0i \\ 2j \end{pmatrix}, \text{ and } \mathbf{J}_2 = \begin{pmatrix} -\sin(\theta_1=0) \\ \cos(\theta_1=0) \end{pmatrix} = \begin{pmatrix} 0i \\ j \end{pmatrix}$$

both joints generate endpoint velocity along the y-axis, thus, the motion is restricted along vertical direction.

- When the arm is flexed (θ_2≈180° B, and C positions)

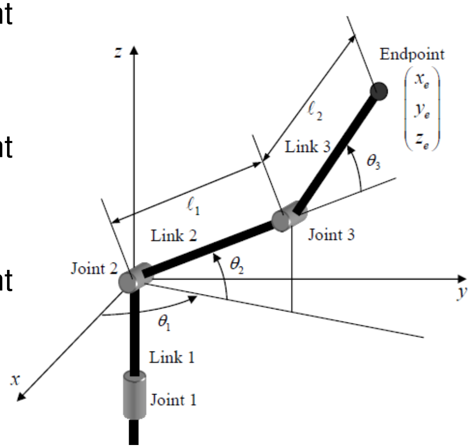
$$\mathbf{J}_1 \approx \begin{pmatrix} -((l_1-l_2)\approx 0)\sin\theta_1 \\ ((l_1-l_2)\approx 0)\cos\theta_1 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{and } \mathbf{J}_2 \approx \begin{pmatrix} l_2\sin\theta_1 \\ -l_2\cos\theta_1 \end{pmatrix}$$

First joint does not produce any contribution to endpoint motion

$$\mathbf{J}(\theta_1,\theta_2) = \begin{pmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{pmatrix}$$

Jacobian of a 3-Link Arm

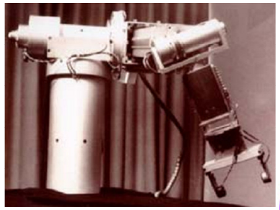
- Lock joint 2 and 3, move joint 1 with unit ang rate and find endpoint velocity → [J_1]
- Lock joint 1 and 3, move joint 2 with unit ang rate and find endpoint velocity → [J_2]
- Lock joint 1 and 2, move joint 3 with unit ang rate and find endpoint velocity → [J_3]
- Determine Jacobian as
  J = [J_1 J_2 J_3]
- Find singularities by |J|=0

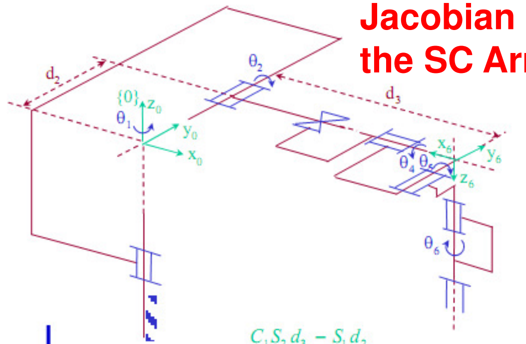


Singularity and Redundancy

- Sometimes, such singular configurations exist in the middle of the workspace seriously degrading mobility and manipulatability of the robot
- To overcome this difficulty, endpoint trajectories can be planned away from singular configurations. Alternatively, additional degrees of freedom should be included so that even when some degrees of freedom are lost at certain configurations, the robot can still maintain an adequate number of degrees of freedom (Redundant Manipulators).
- To locate the endpoint at any position with any orientation, a planner manipulator needs 3 variables (x_e, y_e, ϕ_e), whereas a spacial manipulator needs 6 variables ($x_e, y_e, z_e, \phi_x, \phi_y, \phi_z$). Same number of degrees of freedom are required for non-redundent planner and special arms.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6





Jacobian of the SC Arm

$$x = \begin{pmatrix} x_P \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}_{12\times1} = \begin{vmatrix} C_1S_2d_3 - S_1d_2 \\ S_1S_2d_3 + C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix}$$

SCA Position Jacobian

Position

$$x_p = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \end{bmatrix}$$

By Partial Differentiation

$$\dot{x}_p = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} -s_1s_2d_3 - c_1d_2 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ c_1s_2d_3 - s_1d_2 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

Position Jacobian

$$\dot{x}_{p(3\times1)} = J_{x_p(3\times6)}(q)\,\dot{q}_{(6x1)}$$

Linear Velocity **V**

SCA Orientation Jacobian

Orientation: Direction Cosines

$$x_R = \begin{bmatrix} r_1(q)_{3\times1} \\ r_2(q)_{3\times1} \\ r_3(q)_{3\times1} \end{bmatrix}_{9\times1}$$

$$\underset{9\times1}{\dot{x}_R} = \underset{9\times6}{J_{X_R}}(q)\underset{6\times1}{\dot{q}}$$

$$\begin{pmatrix} r_{1x} \\ r_{1y} \\ r_{1z} \end{pmatrix} = \begin{pmatrix} C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \end{pmatrix}$$

$$\begin{pmatrix} r_{2x} \\ r_{2y} \\ r_{2z} \end{pmatrix} = \begin{pmatrix} C_1[-C_2(C_4C_5C_6 - S_4C_6) - S_2S_5S_6] - S_1(-S_4C_5C_6 + C_4C_6) \\ S_1[-C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(-S_4C_5C_6 + C_4C_6) \\ S_2(C_4C_5S_6 - S_4S_6) - C_2S_5S_6 \end{pmatrix}$$

$$\begin{pmatrix} r_{3x} \\ r_{3y} \\ r_{3z} \end{pmatrix} = \begin{pmatrix} C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4C_5 + C_2C_5 \end{pmatrix}$$

Stanford Schinman Arm

By Partial Differentiation

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9x1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1}_{3\times1} & \cdots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1}_{3\times1} & \cdots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1}_{3\times1} & \cdots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9x6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6x1)}$$

Representations

$$X = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

x_P:
- Cartesian 3×1
- Spherical 3×1
- Cylindrical 3×1
- ….

x_R:
- Euler Angles 3×1
- Direction Cosines 9×1
- Euler Parameters 4×1

Total Jacobian

$$\dot{x}_P = J_{X_P}(q)\dot{q}$$
$$\dot{x}_R = J_{X_R}(q)\dot{q}$$

$$\begin{pmatrix} \dot{x}_P \\ \dot{x}_R \end{pmatrix} = \begin{pmatrix} J_{X_P}(q) \\ J_{X_R}(q) \end{pmatrix}\dot{q}$$

Cartesian & Direction Cosines

$$\dot{X}_{(12x1)} = J_X(q)_{(12x6)}\,\dot{q}_{(6x1)}$$

Jacobian depends on the representation and arm configuration